Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-11-29

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to warm-up questions

Question 1. If we apply Green's Theorem, the integrals become

$$-\iint_D 0\,\mathrm{d}x\,\mathrm{d}y, \quad -\iint_D 2\,\mathrm{d}x\,\mathrm{d}y, \quad -\iint_D -1\,\mathrm{d}x\,\mathrm{d}y.$$

Notice the minus sign because C is oriented clockwise. So the first integral is zero, the second is -2 times the area of D, and the third is the area of D.

Question 2. This is a cone. You could solve this problem by finding a Cartesian equation for it: $x^2+y^2=z^2$. Then a normal vector to the tangent plane at the point (x,y,z) is $\langle 2x, 2y, -2z\rangle$. A normal vector to the xy-plane is $\langle 0,0,1\rangle$, so we just have to evaluate the angle between these:

$$\cos\theta = \frac{-2z}{2\sqrt{x^2+y^2+z^2}} = \frac{-2z}{2\sqrt{2z^2}} = -1/\sqrt{2}$$

so $\theta = 3\pi/4$. The angle between the two planes should be $\leq \pi/2$ though, so the correct answer is $\pi - \theta = \boxed{\pi/4}$ (why?).

Alternatively we could compute a normal vector from the parametrization via $\mathbf{r}_\theta \times \mathbf{r}_z$, and use that in place of $\langle 2x, 2y, -2z$ in the preceding problem. Try it and check that you get the same answer.

Question 3. No. The curl of $\langle P, Q, 0\rangle$ is $\langle -Q_z, P_z, Q_x - P_y\rangle$, so if P or Q depends on z then the curl may not be vertical.

Answers to computations

For some problems I will only provide an outline of the solution.

Problem 1.

(a) $\mathbf{F} = \langle P(x), Q(y), R(z)\rangle$. It's important that you report your answer like this, and not just $\langle P, Q, R\rangle$.

(b) The answer is zero, because $\nabla\times\mathbf{F} = \nabla\times(\nabla f) = \mathbf{0}$. Sure, you could apply Stokes' if you want, but that would just say

$$\int_{\partial S}(\nabla\times\mathbf{F})\cdot\mathrm{d}\mathbf{r} = \iint_S(\nabla\times(\nabla\times\mathbf{F}))\cdot\mathrm{d}\mathbf{S}$$

which is zero for the same reason. Make sure you don't make the mistake of rewriting the integral as $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$; that's not how Stokes' works.

Problem 2. Apply the Divergence Theorem to obtain the equivalent integral

$$\iiint_E (1-2x^2-4y^2-8z^2)\,\mathrm{d}V$$

where E is the 3D region enclosed by S. If we want to maximize this integral, then E should be precisely the region on which $1-2x^2-4y^2-8z^2 \geq 0$. That means S is the surface $1-2x^2-4y^2-8z^2=0$, which is an ellipsoid.

Problem 3. Use the Divergence Theorem as follows, where E denotes the region enclosed by S.

$$\iiint_E \mathrm{d}V = \iint_S \frac{1}{3}\langle x,y,z\rangle\cdot\mathrm{d}\mathbf{S}$$

Then $\mathrm{d}\mathbf{S}$ will be $\pm\mathbf{r}_s\times\mathbf{r}_t\,\mathrm{d}s\,\mathrm{d}t$. We can't tell whether it's $+$ or $-$ for the outwards orientation with the information given, but since the volume needs to be ≥ 0, we can just take either one and put absolute values around our expression. If we do that, then we end up with the formula in the problem.

If you want to compute the volume enclosed by a parametric surface in practice, you will probably take one of $\langle x,0,0\rangle$, $\langle 0,y,0\rangle$ or $\langle 0,0,z\rangle$ in place of $\frac{1}{3}\langle x,y,z\rangle$ when applying the Divergence Theorem.

Problem 4.

(a) $x = t\cos(8\pi t)$ and $y = t\sin(8\pi t)$ for $0 \leq t \leq 1$.

(b) You can either do this by direct parametrization, or using FTLI noting that $\nabla\left(\frac{1}{2}(x^2+y^2)\right) = \langle x, y\rangle$. The latter approach is faster.